MATH 5061 Problem Set 5[1]
Due date: Apr 15, 2024

Problems: (Please hand in your assignments by submitting your PDF via email. **Late submissions will not be accepted.**)

Throughout this assignment, we use (M, g) to denote a smooth n-dimensional Riemannian manifold with its Levi-Civita connection ∇ unless otherwise stated. The Riemann curvature tensor (as a $(0,4)$-tensor) of (M, g) is denoted by R.

1. Prove that the upper half plane $\mathbb{R}^2_+ := \{(x,y) \in \mathbb{R}^2 : y > 0\}$ with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$ is complete.

2. Let (M^n, g) be a complete Riemannian manifold. Suppose there exists constants $a > 0$ and $c \geq 0$ such that for all pairs of points p, q in M, and for all minimizing geodesics $\gamma(s)$, which is parametrized by arc length, joining p to q, we have
$$\mathrm{Ric}(\gamma'(s), \gamma'(s)) \geq a + \frac{df}{ds} \qquad \text{along } \gamma,$$
where f is a functions of s such that $|f(s)| \leq c$ along γ. Prove that (M^n, g) is compact.

3. Let (M^n, g) be a complete Riemannian manifold with non-positive sectional curvature, i.e. $K \leq 0$. Show that any homotopy class of paths with fixed end points p and q in M contains a unique geodesic.

4. Show that any even dimensional complete manifold with constant positive sectional curvature is isometric to either $\mathbb{S}^{2n}$ or $\mathbb{RP}^{2n}$,equipped with the canonical round metric.

5. Using the identification $\mathbb{C}^2 \cong \mathbb{R}^4$, we denote the unit sphere by $\mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Let $h : \mathbb{S}^3 \to \mathbb{S}^3$ be the smooth map given by
$$h(z_1, z_2) = (e^{\frac{2\pi}{q} i} z_1, e^{\frac{2\pi r}{q} i} z_2)$$
where q and r are relatively prime integers with $q > 2$.
   (a) Show that $G = \{\mathrm{id}, h, \cdots, h^{q-1}\}$ is a group of isometries of the sphere $\mathbb{S}^3$ with the standard round metric. Prove that the quotient space $\mathbb{S}^3/G$ is a smooth manifold. This is called a *lens space*.
   (b) Suppose the lens space $\mathbb{S}^3/G$ is equipped with the natural Riemannian metric such that the projection map $\pi : \mathbb{S}^3 \to \mathbb{S}^3/G$ is a local isometry. Show that all the geodesics of $\mathbb{S}^3/G$ are closed but can have different lengths.

[1] Last revised on March 24, 2024